Black-Scholes equation.

Our next goal is to find the price of a Call/Put option. We will consider the Call option, the function for the Put option can be obtained using the Put-Call parity eq-n, once the price of the Call option is known.

Recall: if B is the balance in a bank at a risk-free rate r, then $\frac{dB}{B} = r\,dt$.

The price S of a stock satisfies the diff. eq-n: $\frac{dS}{S} = \mu\,dt + \sigma\,dB(t)$ (*), where $B(t)$ is the standard Brownian motion.

Rmk/defn. The parameter μ is called the mean growth rate and σ is called volatility.

Let $C(S,t)$ be the price of a Call option for stock S at time t. Using Ito's formula (see page) and eq-n (*) above, we get

$$dC(S,t) = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}\cdot dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 =$$

$$= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(\mu S dt + \sigma S\, dB(t)) + \frac{\partial^2 C}{\partial S^2}\left(\frac{1}{2}\sigma^2 S^2 dB(t)^2 + O(dt)\right)$$

$$= \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \cdot \frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S \frac{\partial C}{\partial S}\, dB(t) \ (\mathrm{I})$$

On the other hand, we can represent $C(S,t)$ as a linear combination of S and B, where the coefficients are functions in S and t:

$$C(S,t) = u(S,t)S - v(S,t)B$$

This expression gives

$$dC(S,t) = u\,dS + S\,du - B\,dv - v\,dB.$$

Notice that you need to borrow more money to buy more stock, i.e. $S\,du = B\,dv$, hence

$$dC(S,t) = u\,dS - v\,dB = u(\mu S dt + \sigma S\, dB(t)) -$$
$$- v r B dt \ (\mathrm{II}).$$

Equating I and II, we get:

coeffients of $dB(t)$: $u \sigma S = \sigma S \frac{\partial C}{\partial S}$, as $\sigma \neq 0$ and $S \neq 0$, $u = \frac{\partial C}{\partial S}$;

coefficients of dt:

$$\boxed{\frac{\partial C}{\partial t} = -\frac{\sigma^2 S^2}{2} \cdot \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC.}$$

This equation is called the <u>Black-Scholes differential equation.</u>

Example. Suppose $dS = 5dt + 6S^2 dB(t)$. Find the corresponding Black-Scholes eq-n.

Sol-n: using Ito's formula, we get

$$dC(S,t) = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\cdot\frac{\partial^2 C}{\partial S^2}dS^2 =$$

$$= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(5dt + 6S^2 dB(t)) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\cdot 36S^4 \overset{dt}{\overset{\shortparallel}{dB(t)^2}} =$$

$$= \left(\frac{\partial C}{\partial t} + 5\cdot\frac{\partial C}{\partial S} + 18S^4\frac{\partial^2 C}{\partial S^2}\right)dt + 6S^2\frac{\partial C}{\partial t}dB(t) \quad (*).$$

Using the expression $C(S,t) = uS - vB$, obtain

$$dC(S,t) = u\,dS - v\,dB = u(5dt + 6S^2 dB(t)) - vrB\,dt =$$

$$= (5u - vrB)dt + 6S^2u\,dB(t). \quad (**)$$

As $(*) = (**)$:

coeff-s of $dB(t)$: $u = \frac{\partial C}{\partial S}$ which, using that $v = \frac{uS - C(S,t)}{B}$, gives $v = \frac{\frac{\partial C}{\partial S}S - C(S,t)}{B}$.

coeff-s of dt: $\frac{\partial C}{\partial t} + \cancel{5\cdot\frac{\partial C}{\partial S}} + 18\cdot\frac{S^4\partial^2 C}{\partial S^2} = \cancel{\frac{5\cdot\partial C}{\partial S}} - rS\cdot\frac{\partial C}{\partial S} + rC$ or

$$\boxed{\frac{\partial C}{\partial t} + 18S^4\cdot\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + rC = 0}$$

Fact. There is a change of variables after which the Black-Scholes equation becomes the heat equation:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

Heat Equation.

Consider a thin (one-dimensional) bar. The solution of the heat eq-n identifies the temperature of the bar at location x at time t. We also need to know the initial temperature distribution (initial condition), i.e. the f-n $u(x,0) := \psi(x)$.

We need two more conditions:

$u(l,t) = \chi_l(t)$ and $u(r,t) = \chi_r(t)$.

What is the solution of the heat eq-n?

Answer: $u(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{(s-x)^2}{2t}} ds.$

Exercise: check it for $\varphi(x) = \text{const}$.

Rmk: the function $u(x,t)$ for $\varphi(x) \equiv 1$ is the cdf for the distr-n $N(x, t)$.

Sol-ns of the Black-Scholes Eq-n.

Recall that the initial condition (value at time $t = T$) for the call option is

$$C_E(S,T) = \max\{0, S-E\}.$$

The sol-n is given by

$C_E(S,t) = S \cdot \varphi(d_1) - E e^{-r(T-t)} \varphi(d_2)$, where

$\varphi(d)$ is the cdf of $N(0,1)$ $\left(\varphi(d) = \int_{-\infty}^{d} e^{-x^2/2} dx\right)$;

$$d_1 = \frac{\ln(S/E) + (T-t)(r + \sigma^2/2)}{\sigma\sqrt{T-t}};$$

$$d_2 = \frac{\ln(S/E) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

Analysis / Interpretations

① Consider the limit $t \to T^-$ (from the left):

(a) $S > E$: $\ln(S/E) > 0$, so $\dfrac{\ln(S/E)}{\sigma\sqrt{T-t}} \to +\infty$ and

$\dfrac{(T-t)(r+\sigma^2/2)}{\sigma\sqrt{T-t}} \to 0$ (independent of the comparison of S to E), hence, $d_1, d_2 \to +\infty$ with $\Phi(d_1) = \Phi(d_2) = 1$,

so $C_E(S,T) = S - E$.

(b) $S < E$: $\ln(S/E) < 0$ and $\dfrac{\ln(S/E)}{\sigma\sqrt{T-t}} \to -\infty$

with $d_1, d_2 \to -\infty$ and $\Phi(d_1) = \Phi(d_2) = 0$, so

$C_E(S,T) = 0$.

(c) $S = E$: $\ln(S/E) = \ln(1) = 0$ and $d_1 = d_2 = 0$ with

$\Phi(d_1) = \Phi(d_2) = 1/2$, giving $C_E(S,T) = \frac{1}{2}S - \frac{1}{2}E = 0$.

② $S \to \infty$: $d_1 = d_2 \to \infty$, $\Phi(d_1) = \Phi(d_2) = 1$ with

$C_E(S,t) = S - E = S$ (E is negligible)

③ $S \to 0$: $d_1 = d_2 \to -\infty$ with $\Phi(d_1) = \Phi(d_2) = 0$ and

$C_E(S,t) = 0$.

Rmk. As we already mentioned, the boundary condition for Black-Scholes eq-n for Call options corresponding to $\Psi(x)$ (for the heat eq-n) is $C_E(S,T) = \max(S-E, 0)$. This can be easily seen to be equivalent to the limit in 1.

Notice that $\Delta S = \mu S \Delta t + \sigma S \Delta B(t)$, when $S \to 0$ gives $\Delta S \sim 0$, giving the boundary condition $C_E(0,t) = 0$, which agrees with 3

1, 2 and 3 form a complete set of boundary conditions and correspond to $\Psi(x)$, $X_\ell(t)$ and $X_r(t)$ (boundary cond-ns for the heat eq-n with bar of infinite length, i.e. $\ell = -\infty$, $r = +\infty$).

Prices of stocks of some companies over last 5 years